Comprehensive Portfolio Optimization: From Fundamentals to the Tangency Portfolio

Nikhil

June 20, 2025

Introduction to Portfolio Optimization

Portfolio optimization is a cornerstone of modern finance, guiding investors on how to combine various assets to achieve desired risk and return objectives. The central tenet is that combining assets strategically can reduce overall portfolio risk without necessarily sacrificing expected returns, a concept known as diversification. This document will build from foundational concepts of individual asset risk and return, progressing through simple two-asset portfolios, and culminating in the derivation and importance of the Tangency Portfolio for multiple risky assets when a risk-free asset is available.

Fundamentals of Risk and Return for Individual Assets

Before combining assets, it's crucial to understand how we quantify their individual characteristics.

Expected Return (Arithmetic Mean) ($\bar{r}$)

The expected return of an asset, often estimated by its arithmetic mean, is the average of all observed returns over a period. If we have N historical observations of return r_i for an asset:

$$\bar{r} = \frac{1}{N}\sum_{i=1}^{N} r_i$$

Variance (σ^2) and Standard Deviation (σ)

These measures quantify the risk or volatility of an asset's returns, indicating how much the returns tend to deviate from their expected value. For a sample of N observations, the sample variance uses $N-1$ in the denominator for an unbiased estimate.

- **Variance (σ^2):** The average of the squared differences from the mean.

$$\sigma^2 = \frac{1}{N-1}\sum_{i=1}^{N}(r_i - \bar{r})^2$$

- **Standard Deviation (σ):** The square root of the variance. It's often preferred as a risk measure because it's in the same units as the expected return, making it easier to interpret.

$$\sigma = \sqrt{\sigma^2}$$

Properties of Variance

- **Non-negativity:** $\text{Var}(X) \geq 0$. Risk cannot be negative.
- **Scaling Property:** If a is a constant, then $\text{Var}(aX) = a^2\text{Var}(X)$.
- **Variance of a Constant:** If c is a constant, then $\text{Var}(c) = 0$.

Covariance (σ_{XY})

Covariance measures the extent to which two asset returns (X and Y) move together. For a sample of N paired observations:

$$\sigma_{XY} = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X})(Y_i - \bar{Y})$$

Properties of Covariance

- **Symmetry:** $\text{Cov}(X, Y) = \text{Cov}(Y, X)$.
- **Covariance of an Asset with Itself:** $\text{Cov}(X, X) = \text{Var}(X)$.
- **Covariance with a Constant:** $\text{Cov}(X, c) = 0$.

Correlation Coefficient (ρ_{XY})

The correlation coefficient is a standardized version of covariance, ranging from -1 (perfect negative correlation) to +1 (perfect positive correlation).

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

This also implies that $\sigma_{XY} = \rho_{XY}\sigma_X\sigma_Y$.

Portfolio of Two Risky Assets: The Portfolio Frontier

When combining two risky assets with weights w_1 and w_2 ($w_1 + w_2 = 1$, $0 \leq w_i \leq 1$), the portfolio's expected return and variance are:

$$\bar{r}_p = w_1\bar{r}_1 + w_2\bar{r}_2$$

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}$$

The Portfolio Frontier and Efficient Frontier

When we plot the expected return against standard deviation for all possible combinations of two risky assets, we trace out a curve called the **portfolio frontier**.

Choosing the Optimal Portfolio (Two Risky Assets Only)

The choice depends on individual risk tolerance. Investors will select a portfolio on the efficient frontier that aligns with their desired balance of risk and return.

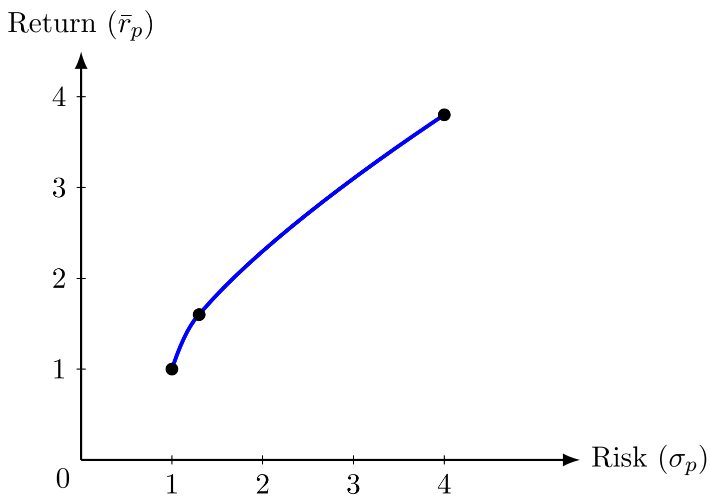

Figure 1: The Portfolio Frontier for Two Risky Assets

Portfolio of One Risky Asset and One Risk-Free Asset: The Capital Allocation Line (CAL)

When combining a risky asset (R) with a risk-free asset (F) (which has return r_F and zero standard deviation, $\sigma_F = 0$), the portfolio's expected return and standard deviation are:

$$\bar{r}_p = w_R \bar{r}_R + (1 - w_R) r_F$$

$$\sigma_p = w_R \sigma_R$$

Here, w_R is the weight invested in the risky asset, and $(1 - w_R)$ is the weight invested in the risk-free asset. Note that if $w_R > 1$, the investor is borrowing at the risk-free rate to invest more in the risky asset (leverage).

Mathematical Derivation of Linearity

By rearranging the expression for σ_p, we get $w_R = \sigma_p / \sigma_R$. Substituting this into the portfolio return equation:

$$\bar{r}_p = \left(\frac{\sigma_p}{\sigma_R} \right) \bar{r}_R + \left(1 - \frac{\sigma_p}{\sigma_R} \right) r_F$$

$$\bar{r}_p = \frac{\bar{r}_R}{\sigma_R} \sigma_p + r_F - \frac{r_F}{\sigma_R} \sigma_p$$

Rearranging terms, we obtain the equation for the Capital Allocation Line (CAL):

$$\bar{r}_p = r_F + \left(\frac{\bar{r}_R - r_F}{\sigma_R} \right) \sigma_p$$

This equation is linear in σ_p, indicating that combinations of a risk-free asset and a risky asset lie on a straight line. The slope of this line is the Sharpe Ratio of the risky asset.

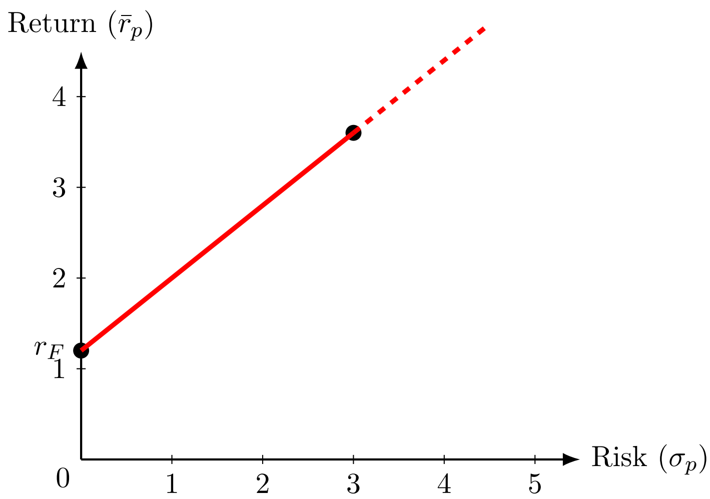

Figure 2: The Capital Allocation Line (CAL)

Portfolio of Multiple Risky Assets with a Risk-Free Asset: The Tangency Portfolio

When an investor has access to multiple risky assets and a risk-free asset, the optimal strategy involves identifying the single "best" portfolio of risky assets. This portfolio, known as the Tangency Portfolio, is combined with the risk-free asset to form the investor's overall portfolio.

Matrix Notation for Multiple Risky Assets

To manage multiple assets efficiently, matrix notation is indispensable. For a portfolio of N risky assets:

- **Weights Vector (w):** An $(N \times 1)$ column vector where w_j is the proportion of total risky wealth invested in asset j.

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix}$$

- **Expected Returns Vector ($\bar{\mathbf{r}}$):** An $(N \times 1)$ column vector of expected returns for each asset.

$$\bar{\mathbf{r}} = \begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \vdots \\ \bar{r}_N \end{pmatrix}$$

- **Expected Portfolio Return:** The expected return of a portfolio is the weighted average of the expected returns of its constituent assets.

$$\bar{r}_p = \mathbf{w}^T \bar{\mathbf{r}}$$

- **Covariance Matrix ($\mathbf{\Sigma}$):** An $(N \times N)$ symmetric matrix containing all individual variances on the diagonal and all pairwise covariances off-diagonal.

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1N} \\ \sigma_{21} & \sigma_2^2 & \dots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \dots & \sigma_N^2 \end{pmatrix}$$

- **Portfolio Variance:** The variance of a portfolio is calculated using the covariance matrix.

$$\sigma_p^2 = \mathbf{w}^T \mathbf{\Sigma} \mathbf{w}$$

The Optimization Problem for the Tangency Portfolio

The Tangency Portfolio (also known as the Optimal Risky Portfolio) is the portfolio of risky assets that yields the highest Sharpe Ratio when combined with the risk-free asset. It is found by minimizing portfolio variance subject to a target expected *excess return* (return above the risk-free rate).

Let $\bar{\mathbf{x}} = \bar{\mathbf{r}} - r_F \mathbf{1}$ be the vector of expected excess returns (where $\mathbf{1}$ is a vector of ones). The optimization problem is:

$$\min_{\mathbf{w}} \quad \mathbf{w}^T \mathbf{\Sigma} \mathbf{w} \quad \text{(Minimize Portfolio Variance)}$$

$$\text{s.t.} \quad \mathbf{w}^T \bar{\mathbf{x}} = m \quad \text{(Subject to a target expected excess return } m\text{)}$$

Lagrangian Formulation

We use the Lagrange Multiplier method to solve this constrained optimization problem. The Lagrangian L combines the objective function and the constraint:

$$L(\mathbf{w}, \lambda) = \mathbf{w}^T \mathbf{\Sigma} \mathbf{w} - 2\lambda(\mathbf{w}^T \bar{\mathbf{x}} - m)$$

The factor 2λ is used for algebraic convenience, simplifying the derivatives.

Detailed First-Order Condition (FOC) for $\frac{\partial L}{\partial \mathbf{w}}$ (2-Stock Example)

To find the minimum point of the Lagrangian, we take the partial derivative of L with respect to the vector of weights $\mathbf{w}$ and set it to zero. Let's explicitly expand this for a two-stock portfolio (Stock A and Stock B) to avoid complex matrix calculus notation initially.

For two stocks:

$$\mathbf{w} = \begin{pmatrix} w_A \\ w_B \end{pmatrix} \quad \bar{\mathbf{x}} = \begin{pmatrix} \bar{x}_A \\ \bar{x}_B \end{pmatrix} \quad \mathbf{\Sigma} = \begin{pmatrix} \sigma_A^2 & \sigma_{AB} \\ \sigma_{BA} & \sigma_B^2 \end{pmatrix}$$

(Recall: $\sigma_{AB} = \sigma_{BA}$)

First, expand the portfolio variance term $\mathbf{w}^T \mathbf{\Sigma} \mathbf{w}$:

$$\mathbf{w}^T \mathbf{\Sigma} \mathbf{w} = \begin{pmatrix} w_A & w_B \end{pmatrix} \begin{pmatrix} \sigma_A^2 & \sigma_{AB} \\ \sigma_{BA} & \sigma_B^2 \end{pmatrix} \begin{pmatrix} w_A \\ w_B \end{pmatrix}$$

$$= \begin{pmatrix} w_A & w_B \end{pmatrix} \begin{pmatrix} \sigma_A^2 w_A + \sigma_{AB} w_B \\ \sigma_{BA} w_A + \sigma_B^2 w_B \end{pmatrix}$$

$$= w_A(\sigma_A^2 w_A + \sigma_{AB} w_B) + w_B(\sigma_{BA} w_A + \sigma_B^2 w_B)$$
$$= w_A^2\sigma_A^2 + w_A w_B \sigma_{AB} + w_B w_A \sigma_{BA} + w_B^2\sigma_B^2$$

Since $\sigma_{AB} = \sigma_{BA}$:

$$= w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B \sigma_{AB}$$

This is the familiar expanded formula for two assets.

Now, differentiate the Lagrangian $L = (w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B\sigma_{AB}) - 2\lambda(w_A\bar{x}_A + w_B\bar{x}_B - m)$ with respect to each weight separately:

Partial Derivative with respect to w_A ($\frac{\partial L}{\partial w_A}$)

$$\frac{\partial L}{\partial w_A} = \frac{\partial}{\partial w_A}(w_A^2\sigma_A^2) + \frac{\partial}{\partial w_A}(w_B^2\sigma_B^2) + \frac{\partial}{\partial w_A}(2w_A w_B\sigma_{AB}) - \frac{\partial}{\partial w_A}(2\lambda w_A\bar{x}_A) - \frac{\partial}{\partial w_A}(2\lambda w_B\bar{x}_B - 2\lambda m)$$

Evaluating each term:

- $\frac{\partial}{\partial w_A}(w_A^2\sigma_A^2) = 2w_A\sigma_A^2$
- $\frac{\partial}{\partial w_A}(w_B^2\sigma_B^2) = 0$ (as w_B is treated as a constant for this partial derivative)
- $\frac{\partial}{\partial w_A}(2w_A w_B\sigma_{AB}) = 2w_B\sigma_{AB}$
- $\frac{\partial}{\partial w_A}(-2\lambda w_A\bar{x}_A) = -2\lambda\bar{x}_A$
- The last terms are constants with respect to w_A, so their derivative is 0.

Setting the sum of derivatives to zero:

$$2w_A\sigma_A^2 + 2w_B\sigma_{AB} - 2\lambda\bar{x}_A = 0$$

Divide by 2:

$$w_A\sigma_A^2 + w_B\sigma_{AB} = \lambda\bar{x}_A$$

Partial Derivative with respect to w_B ($\frac{\partial L}{\partial w_B}$)

Similarly, evaluating each term with respect to w_B:

- $\frac{\partial}{\partial w_B}(w_A^2\sigma_A^2) = 0$
- $\frac{\partial}{\partial w_B}(w_B^2\sigma_B^2) = 2w_B\sigma_B^2$
- $\frac{\partial}{\partial w_B}(2w_A w_B\sigma_{AB}) = 2w_A\sigma_{AB}$
- $\frac{\partial}{\partial w_B}(-2\lambda w_B\bar{x}_B) = -2\lambda\bar{x}_B$

Setting the sum of derivatives to zero:

$$2w_B\sigma_B^2 + 2w_A\sigma_{AB} - 2\lambda\bar{x}_B = 0$$

Divide by 2:

$$w_A\sigma_{AB} + w_B\sigma_B^2 = \lambda\bar{x}_B$$

Combining Results in Matrix Form

These two linear equations can be written compactly in matrix form:

$$\begin{pmatrix} \sigma_A^2 & \sigma_{AB} \\ \sigma_{AB} & \sigma_B^2 \end{pmatrix} \begin{pmatrix} w_A \\ w_B \end{pmatrix} = \lambda \begin{pmatrix} \bar{x}_A \\ \bar{x}_B \end{pmatrix}$$

Which is precisely:

$$\boldsymbol{\Sigma}\mathbf{w} = \lambda\bar{\mathbf{x}}$$

To solve for the weight vector $\mathbf{w}$, we pre-multiply both sides by the inverse of the covariance matrix, $\boldsymbol{\Sigma}^{-1}$:

$$\mathbf{w} = \lambda\boldsymbol{\Sigma}^{-1}\bar{\mathbf{x}}$$

Insight: This detailed step-by-step differentiation for two assets clearly shows how the general matrix derivative result $\frac{\partial}{\partial \mathbf{w}}(\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}) = 2\boldsymbol{\Sigma}\mathbf{w}$ arises. It consolidates the individual partial derivatives into a compact and general matrix form, which is scalable for any number of assets.

Determining λ for the Tangency Portfolio

The Tangency Portfolio is a specific optimal risky portfolio. Its key characteristic is that its weights *within the risky asset component* sum to 1. That is, $\mathbf{w}^T\boldsymbol{i} = 1$, where $\boldsymbol{i}$ is a vector of ones. We use this constraint to find the specific value of λ that corresponds to the tangency portfolio.

Substitute $\mathbf{w}_T = \lambda\boldsymbol{\Sigma}^{-1}\bar{\mathbf{x}}$ into the sum-to-one constraint:

$$(\lambda\boldsymbol{\Sigma}^{-1}\bar{\mathbf{x}})^T\boldsymbol{i} = 1$$

Since λ is a scalar, we can pull it out. Also, the transpose of a product of matrices is the product of their transposes in reverse order: $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$. And for a symmetric matrix like $\boldsymbol{\Sigma}^{-1}$, its transpose is itself, $(\boldsymbol{\Sigma}^{-1})^T = \boldsymbol{\Sigma}^{-1}$.

$$\lambda(\bar{\mathbf{x}}^T(\boldsymbol{\Sigma}^{-1})^T\boldsymbol{i}) = 1$$

$$\lambda(\bar{\mathbf{x}}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{i}) = 1$$

Solving for λ:

$$\lambda = \frac{1}{\bar{\mathbf{x}}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{i}}$$

Final Weights for the Tangency Portfolio ($\mathbf{w}_T$)

Substitute this specific value of λ back into the expression for $\mathbf{w}$:

$$\mathbf{w}_T = \frac{1}{\bar{\mathbf{x}}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{i}}\boldsymbol{\Sigma}^{-1}\bar{\mathbf{x}}$$

This formula directly provides the vector of weights for each risky asset that constitutes the Tangency Portfolio. These weights, by construction, sum to 1.

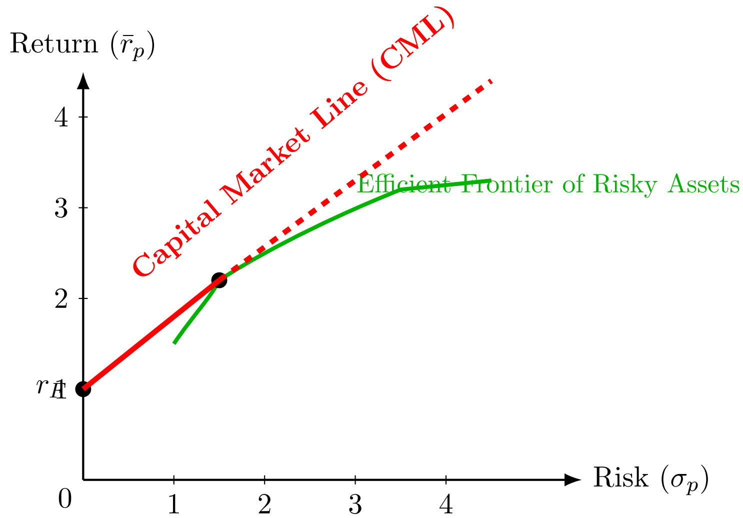

Figure 3: The Capital Market Line (CML) and Tangency Portfolio

The Importance of the Tangency Portfolio and the Capital Market Line (CML)

The Tangency Portfolio is a cornerstone of modern portfolio theory, particularly when a risk-free asset is available. It is the crucial point where the Capital Allocation Line (CAL) becomes the Capital Market Line (CML).

The Capital Market Line (CML):

- The CML is a special Capital Allocation Line. It represents the best possible risk-return trade-off that an investor can achieve by combining the risk-free asset with the *optimal* risky portfolio.
- This optimal risky portfolio is the Tangency Portfolio (T), which is the point where the CAL originating from the risk-free rate becomes tangent to the efficient frontier of risky assets.
- Any portfolio on the CML is a combination of the risk-free asset and the Tangency Portfolio. Investors choose their position on the CML based on their risk tolerance.

Importance of the Tangency Rule (Separation Theorem)

The concept of the Tangency Portfolio is fundamental because it leads to the groundbreaking "Separation Theorem" (also known as the Two-Fund Separation Theorem).

- **Universally Optimal Risky Portfolio:** The Tangency Portfolio is the *single optimal portfolio of risky assets* for *all* investors, regardless of their individual risk aversion. It provides the highest possible Sharpe Ratio among all possible risky asset combinations.
- **Separation of Investment Decisions:** The theorem states that an investor's portfolio decision can be conceptually separated into two distinct, independent steps:
  1. **Step 1: Determine the Optimal Risky Portfolio (Technical Decision):** This involves analyzing the expected returns, variances, and covariances of all available risky assets to identify the Tangency Portfolio. This is a purely objective, mathematical exercise.

2. **Step 2: Allocate Between Risk-Free and Optimal Risky Portfolio (Personal Decision):** Based on their individual risk tolerance, investors then decide how much capital to allocate to the risk-free asset and how much to the Tangency Portfolio (or even borrow at the risk-free rate to leverage it).

- **Efficiency and Simplicity:** This separation greatly simplifies the investment process. Instead of searching for a unique optimal portfolio for each individual, all rational investors in the market aim for the same Tangency Portfolio for their risky investments, then tailor their overall risk exposure with the risk-free asset.

Examples for Practice

Example 1: Portfolio of Two Risky Assets

Consider a portfolio made up of shares in Company X and Company Y.

- Company X: Expected Return = 10%, Standard Deviation = 20%
- Company Y: Expected Return = 18%, Standard Deviation = 35%
- Correlation between X and Y ($\rho_{X,Y}$) = 0.5

Question: Calculate the expected return and standard deviation for a portfolio that invests 60% in Company X and 40% in Company Y.

Solution to Example 1

1. **Calculate Covariance ($\sigma_{X,Y}$):** $\sigma_{X,Y} = \rho_{X,Y}\sigma_X\sigma_Y = 0.5 \times 0.20 \times 0.35 = 0.5 \times 0.07 = 0.035$
2. **Calculate Portfolio Expected Return ($\bar{r}_p$):** $w_X = 0.60$, $w_Y = 0.40$ $\bar{r}_p = w_X\bar{r}_X + w_Y\bar{r}_Y$ $\bar{r}_p = (0.60 \times 0.10) + (0.40 \times 0.18)$ $\bar{r}_p = 0.06 + 0.072 = 0.132 = \mathbf{13.2\%}$
3. **Calculate Portfolio Variance (σ_p^2):** $\sigma_p^2 = w_X^2\sigma_X^2 + w_Y^2\sigma_Y^2 + 2w_Xw_Y\sigma_{X,Y}$ $\sigma_p^2 = (0.36)(0.04) + (0.16)(0.1225) + (0.48)(0.035)$ $\sigma_p^2 = 0.0144 + 0.0196 + 0.0168 = 0.0508$
4. **Calculate Portfolio Standard Deviation (σ_p):** $\sigma_p = \sqrt{0.0508} \approx 0.22539 = \mathbf{22.54\%}$

Answer: For a portfolio with 60% in Company X and 40% in Company Y, the expected return is 13.2% and the standard deviation is approximately 22.54%.

Example 2: Optimal Portfolio with Multiple Risky Assets and One Risk-Free Asset

An investor is considering allocating funds between a Risk-Free Asset, a Bond Fund (BF), and a Stock Fund (SF).

Asset Class	Expected Return	Standard Deviation
Risk-Free Asset	4%	0%
Bond Fund (BF)	7%	8%
Stock Fund (SF)	14%	22%

The **correlation coefficient** between the Bond Fund and the Stock Fund is $\rho_{BF,SF} = 0.1$.

Question 1: What are the weights of the Bond Fund and Stock Fund in the Tangency Portfolio?
Question 2: Calculate the Sharpe Ratios for the Bond Fund, Stock Fund, and the Tangency Portfolio.

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Given Information:

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Risk-Free Asset	4%	0%
Bond Fund (BF)	7%	8%
Stock Fund (SF)	14%	22%

The correlation coefficient between the Bond Fund and the Stock Fund is $\rho_{BF,SF} = 0.1$.

We will directly apply the formula for the Tangency Portfolio weights:

$$\mathbf{w}_T = \frac{1}{\bar{x}^T \Sigma^{-1} \mathbf{i}} \Sigma^{-1} \bar{x}$$

Required Components:

- Expected Excess Returns vector ($\bar{x}$): * $E_{BF} = \bar{r}_{BF} - r_f = 0.07 - 0.04 = 0.03$ * $E_{SF} = \bar{r}_{SF} - r_f = 0.14 - 0.04 = 0.10$
$$\bar{x} = \begin{pmatrix} 0.03 \\ 0.10 \end{pmatrix}$$
- Inverse of the Covariance Matrix (Σ^{-1}): First, we calculate the covariance matrix (Σ):
$$\Sigma = \begin{pmatrix} \sigma_{BF}^2 & \rho_{BF,SF}\sigma_{BF}\sigma_{SF} \\ \rho_{BF,SF}\sigma_{BF}\sigma_{SF} & \sigma_{SF}^2 \end{pmatrix} = \begin{pmatrix} (0.08)^2 & 0.1 \times 0.08 \times 0.22 \\ 0.1 \times 0.08 \times 0.22 & (0.22)^2 \end{pmatrix} = \begin{pmatrix} 0.0064 & 0.00176 \\ 0.00176 & 0.0484 \end{pmatrix}$$
Then, we invert it: Determinant$(\Sigma) = (0.0064)(0.0484) - (0.00176)^2 = 0.00030976 - 0.0000030976 = 0.0003066624$
$$\Sigma^{-1} = \frac{1}{0.0003066624} \begin{pmatrix} 0.0484 & -0.00176 \\ -0.00176 & 0.0064 \end{pmatrix} \approx \begin{pmatrix} 157.818 & -5.739 \\ -5.739 & 20.871 \end{pmatrix}$$
- Vector of ones ($\mathbf{i}$):
$$\mathbf{i} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Step 1: Calculate the numerator part of the formula: $\Sigma^{-1}\bar{x}$ This represents the unnormalized weights of the risky assets.

$$\Sigma^{-1}\bar{x} = \begin{pmatrix} 157.818 & -5.739 \\ -5.739 & 20.871 \end{pmatrix} \begin{pmatrix} 0.03 \\ 0.10 \end{pmatrix}$$

$$= \begin{pmatrix} (157.818 \times 0.03) + (-5.739 \times 0.10) \\ (-5.739 \times 0.03) + (20.871 \times 0.10) \end{pmatrix}$$

$$= \begin{pmatrix} 4.73454 - 0.5739 \\ -0.17217 + 2.0871 \end{pmatrix} = \begin{pmatrix} 4.16064 \\ 1.91493 \end{pmatrix}$$

Step 2: Calculate the denominator part of the formula: $\bar{x}^T\Sigma^{-1}\mathbf{i}$ This serves as the scaling factor (the sum of the unnormalized weights). First, calculate the product $\Sigma^{-1}\mathbf{i}$:

$$\Sigma^{-1}\mathbf{i} = \begin{pmatrix} 157.818 & -5.739 \\ -5.739 & 20.871 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} (157.818 \times 1) + (-5.739 \times 1) \\ (-5.739 \times 1) + (20.871 \times 1) \end{pmatrix} = \begin{pmatrix} 152.079 \\ 15.132 \end{pmatrix}$$

Now, calculate the dot product $\bar{x}^T(\Sigma^{-1}\mathbf{i})$:

$$\bar{x}^T\Sigma^{-1}\mathbf{i} = \begin{pmatrix} 0.03 & 0.10 \end{pmatrix} \begin{pmatrix} 152.079 \\ 15.132 \end{pmatrix}$$

$$= (0.03 \times 152.079) + (0.10 \times 15.132)$$

$$= 4.56237 + 1.5132 = \mathbf{6.07557}$$

Step 3: Calculate the final Tangency Portfolio Weights (w_T) Now, we combine the results from Step 1 and Step 2 using the formula:

$$\mathbf{w}_T = \frac{1}{\mathbf{6.07557}} \begin{pmatrix} \mathbf{4.16064} \\ \mathbf{1.91493} \end{pmatrix}$$

$$w_{BF} = \frac{4.16064}{6.07557} \approx \mathbf{0.68484} \text{ or } \mathbf{68.48}\%$$

$$w_{SF} = \frac{1.91493}{6.07557} \approx \mathbf{0.31516} \text{ or } \mathbf{31.52}\%$$

These weights indicate the allocation within the risky portfolio (Tangency Portfolio) itself.

Question 2: Calculate the Sharpe Ratios for the Bond Fund, Stock Fund, and the Tangency Portfolio.

The Sharpe Ratio (SR) is calculated as: $SR = \frac{\text{Expected Excess Return}}{\text{Volatility}}$

1. Sharpe Ratio for Bond Fund (SR_{BF}):

$$SR_{BF} = \frac{\bar{r}_{BF} - r_f}{\sigma_{BF}} = \frac{0.07 - 0.04}{0.08} = \frac{0.03}{0.08} = \mathbf{0.375}$$

2. Sharpe Ratio for Stock Fund (SR_{SF}):

$$SR_{SF} = \frac{\bar{r}_{SF} - r_f}{\sigma_{SF}} = \frac{0.14 - 0.04}{0.22} = \frac{0.10}{0.22} \approx \mathbf{0.4545}$$

3. Sharpe Ratio for the Tangency Portfolio (SR_T):

First, we calculate the expected return ($\bar{r}_T$) and standard deviation (σ_T) of the Tangency Portfolio.

- Expected Return of Tangency Portfolio ($\bar{r}_T$):

$$\bar{r}_T = w_{BF}\bar{r}_{BF} + w_{SF}\bar{r}_{SF}$$

$$\bar{r}_T = (0.68484 \times 0.07) + (0.31516 \times 0.14)$$

$$\bar{r}_T = 0.0479388 + 0.0441224 = 0.0920612 \approx \mathbf{0.0921} \text{ or } \mathbf{9.21}\%$$

- Standard Deviation of Tangency Portfolio (σ_T): The standard deviation is calculated using the formula for portfolio volatility, including the covariance term due to correlation:

$$\sigma_T = \sqrt{w_{BF}^2 \sigma_{BF}^2 + w_{SF}^2 \sigma_{SF}^2 + 2 w_{BF} w_{SF} \rho_{BF,SF} \sigma_{BF} \sigma_{SF}}$$

$$\sigma_T = \sqrt{(0.68484)^2 (0.08)^2 + (0.31516)^2 (0.22)^2 + 2(0.68484)(0.31516)(0.1)(0.08)(0.22)}$$

$$\sigma_T = \sqrt{0.0029916 + 0.0048079 + 0.0007606}$$

$$\sigma_T = \sqrt{0.0085601} \approx 0.092520 \approx \mathbf{0.0925} \text{ or } \mathbf{9.25}\%$$

Now, calculate SR_T:

$$SR_T = \frac{\bar{r}_T - r_f}{\sigma_T} = \frac{0.0920612 - 0.04}{0.092520} = \frac{0.0520612}{0.092520} \approx \mathbf{0.5627}$$

Summary of Sharpe Ratios:

- $SR_{BF} = \mathbf{0.375}$
- $SR_{SF} \approx \mathbf{0.4545}$
- $SR_T \approx \mathbf{0.5627}$

The Tangency Portfolio achieves a higher Sharpe Ratio than either individual asset, demonstrating the benefits of diversification.

Example 3: Optimal Portfolio with Multiple Risky Assets and One Risk-Free Asset

An investor is considering allocating funds between a Risk-Free Asset, a Large-Cap Equity Fund (LCE), and a Small-Cap Equity Fund (SCE).

Given Information:

Asset Class	**Expected Return**	**Standard Deviation**
Risk-Free Asset	3%	0%
Large-Cap Equity Fund (LCE)	10%	15%
Small-Cap Equity Fund (SCE)	18%	30%

The correlation coefficient between the Large-Cap Equity Fund and the Small-Cap Equity Fund is $\rho_{LCE,SCE} = 0.4$.

Question 1: What are the weights of the Large-Cap Equity Fund and Small-Cap Equity Fund in the Tangency Portfolio?

Question 2: Calculate the Sharpe Ratios for the Large-Cap Equity Fund, Small-Cap Equity Fund, and the Tangency Portfolio.

Answer Key: Example 3

Given Information (recap):

- Risk-Free Rate (r_f) = 3% (0.03)
- Large-Cap Equity Fund (LCE): Expected Return ($\bar{r}_{LCE}$) = 10% (0.10), Standard Deviation (σ_{LCE}) = 15% (0.15)
- Small-Cap Equity Fund (SCE): Expected Return ($\bar{r}_{SCE}$) = 18% (0.18), Standard Deviation (σ_{SCE}) = 30% (0.30)
- Correlation coefficient ($\rho_{LCE,SCE}$) = 0.4

Answer to Question 1: Weights of LCE and SCE in the Tangency Portfolio

We will directly apply the formula for the Tangency Portfolio weights:

$$\mathbf{w}_T = \frac{1}{\bar{x}^T \Sigma^{-1} \mathbf{i}} \Sigma^{-1} \bar{x}$$

Required Components:

- Expected Excess Returns vector ($\bar{x}$): * $E_{LCE} = \bar{r}_{LCE} - r_f = 0.10 - 0.03 = 0.07$ * $E_{SCE} = \bar{r}_{SCE} - r_f = 0.18 - 0.03 = 0.15$

$$\bar{x} = \begin{pmatrix} 0.07 \\ 0.15 \end{pmatrix}$$

- Inverse of the Covariance Matrix (Σ^{-1}): First, we calculate the covariance matrix (Σ):

$$\Sigma = \begin{pmatrix} \sigma_{LCE}^2 & \rho_{LCE,SCE}\sigma_{LCE}\sigma_{SCE} \\ \rho_{LCE,SCE}\sigma_{LCE}\sigma_{SCE} & \sigma_{SCE}^2 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} (0.15)^2 & 0.4 \times 0.15 \times 0.30 \\ 0.4 \times 0.15 \times 0.30 & (0.30)^2 \end{pmatrix} = \begin{pmatrix} 0.0225 & 0.018 \\ 0.018 & 0.0900 \end{pmatrix}$$

Next, we invert it: Determinant(Σ) $= (0.0225)(0.0900) - (0.018)^2 = 0.002025 - 0.000324 = 0.001701$

$$\Sigma^{-1} = \frac{1}{0.001701} \begin{pmatrix} 0.0900 & -0.018 \\ -0.018 & 0.0225 \end{pmatrix} \approx \begin{pmatrix} 52.91005 & -10.58201 \\ -10.58201 & 13.22751 \end{pmatrix}$$

- Vector of ones ($\mathbf{i}$):

$$\mathbf{i} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Step 1: Calculate the numerator part of the formula: $\Sigma^{-1}\bar{x}$ This represents the unnormalized weights of the risky assets.

$$\Sigma^{-1}\bar{x} = \begin{pmatrix} 52.91005 & -10.58201 \\ -10.58201 & 13.22751 \end{pmatrix} \begin{pmatrix} 0.07 \\ 0.15 \end{pmatrix}$$

$$= \begin{pmatrix} (52.91005 \times 0.07) + (-10.58201 \times 0.15) \\ (-10.58201 \times 0.07) + (13.22751 \times 0.15) \end{pmatrix}$$

$$= \begin{pmatrix} 3.7037035 - 1.5873015 \\ -0.7407407 + 1.9841265 \end{pmatrix} = \begin{pmatrix} 2.116402 \\ 1.2433858 \end{pmatrix}$$

Step 2: Calculate the denominator part of the formula: $\bar{x}^T\Sigma^{-1}\mathbf{i}$ This serves as the scaling factor (the sum of the unnormalized weights). First, calculate the product $\Sigma^{-1}\mathbf{i}$:

$$\Sigma^{-1}\mathbf{i} = \begin{pmatrix} 52.91005 & -10.58201 \\ -10.58201 & 13.22751 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 52.91005 - 10.58201 \\ -10.58201 + 13.22751 \end{pmatrix} = \begin{pmatrix} 42.32804 \\ 2.6455 \end{pmatrix}$$

Now, calculate the dot product $\bar{x}^T(\Sigma^{-1}\mathbf{i})$:

$$\bar{x}^T\Sigma^{-1}\mathbf{i} = \begin{pmatrix} 0.07 & 0.15 \end{pmatrix} \begin{pmatrix} 42.32804 \\ 2.6455 \end{pmatrix}$$

$$= (0.07 \times 42.32804) + (0.15 \times 2.6455)$$

$$= 2.9629628 + 0.396825 = \mathbf{3.3597878}$$

Step 3: Calculate the final Tangency Portfolio Weights (w_T) Now, we combine the results from Step 1 and Step 2 using the formula:

$$\mathbf{w}_T = \frac{1}{\mathbf{3.3597878}} \begin{pmatrix} \mathbf{2.116402} \\ \mathbf{1.2433858} \end{pmatrix}$$

$$w_{LCE} = \frac{2.116402}{3.3597878} \approx \mathbf{0.63004} \text{ or } \mathbf{63.00}\%$$

$$w_{SCE} = \frac{1.2433858}{3.3597878} \approx \mathbf{0.36996} \text{ or } \mathbf{37.00}\%$$

These weights indicate the allocation within the risky portfolio (Tangency Portfolio) itself.

Answer to Question 2: Sharpe Ratios for LCE, SCE, and the Tangency Portfolio

The Sharpe Ratio (SR) is calculated as: $SR = \frac{\text{Expected Excess Return}}{\text{Volatility}}$

1. Sharpe Ratio for Large-Cap Equity Fund (SR_{LCE}):

$$SR_{LCE} = \frac{\bar{r}_{LCE} - r_f}{\sigma_{LCE}} = \frac{0.10 - 0.03}{0.15} = \frac{0.07}{0.15} \approx \mathbf{0.4667}$$

2. Sharpe Ratio for Small-Cap Equity Fund (SR_{SCE}):

$$SR_{SCE} = \frac{\bar{r}_{SCE} - r_f}{\sigma_{SCE}} = \frac{0.18 - 0.03}{0.30} = \frac{0.15}{0.30} = \mathbf{0.5000}$$

3. Sharpe Ratio for the Tangency Portfolio (SR_T):

First, we calculate the expected return ($\bar{r}_T$) and standard deviation (σ_T) of the Tangency Portfolio.

- Expected Return of Tangency Portfolio ($\bar{r}_T$):

$$\bar{r}_T = w_{LCE}\bar{r}_{LCE} + w_{SCE}\bar{r}_{SCE}$$

$$\bar{r}_T = (0.63004 \times 0.10) + (0.36996 \times 0.18)$$

$$\bar{r}_T = 0.063004 + 0.0665928 = 0.1295968 \approx \mathbf{0.1296} \text{ or } \mathbf{12.96\%}$$

- Standard Deviation of Tangency Portfolio (σ_T): The standard deviation is calculated using the formula for portfolio volatility, including the covariance term due to correlation:

$$\sigma_T = \sqrt{w_{LCE}^2\sigma_{LCE}^2 + w_{SCE}^2\sigma_{SCE}^2 + 2w_{LCE}w_{SCE}\rho_{LCE,SCE}\sigma_{LCE}\sigma_{SCE}}$$

$$\sigma_T = \sqrt{(0.63004)^2(0.15)^2 + (0.36996)^2(0.30)^2 + 2(0.63004)(0.36996)(0.4)(0.15)(0.30)}$$

$$\sigma_T = \sqrt{0.008931375 + 0.0123183 + 0.0083995}$$

$$\sigma_T = \sqrt{0.029649175} \approx 0.17219 \approx \mathbf{0.1722} \text{ or } \mathbf{17.22\%}$$

Now, calculate SR_T:

$$SR_T = \frac{\bar{r}_T - r_f}{\sigma_T} = \frac{0.1295968 - 0.03}{0.17219} = \frac{0.0995968}{0.17219} \approx \mathbf{0.5784}$$

Summary of Sharpe Ratios:

- $SR_{LCE} \approx \mathbf{0.4667}$
- $SR_{SCE} = \mathbf{0.5000}$
- $SR_T \approx \mathbf{0.5784}$

As expected, the Tangency Portfolio achieves a higher Sharpe Ratio than either individual risky asset, illustrating the benefits of diversification.